

# THE TAN $2\Theta$ THEOREM FOR INDEFINITE QUADRATIC FORMS

LUKA GRUBIŠIĆ, VADIM KOSTRYKIN, KONSTANTIN A. MAKAROV, AND KREŠIMIR VESELIĆ

ABSTRACT. A version of the Davis-Kahan Tan  $2\Theta$  theorem [SIAM J. Numer. Anal. **7** (1970), 1 – 46] for not necessarily semibounded linear operators defined by quadratic forms is proven. This theorem generalizes a recent result by Motovilov and Selin [Integr. Equat. Oper. Theory **56** (2006), 511 – 542].

## 1. INTRODUCTION

In the 1970 paper [3] Davis and Kahan studied the rotation of spectral subspaces for  $2 \times 2$  operator matrices under off-diagonal perturbations. In particular, they proved the following result, the celebrated “Tan  $2\Theta$  theorem”: Let  $A_{\pm}$  be strictly positive bounded operators in Hilbert spaces  $\mathfrak{H}_{\pm}$ , respectively, and  $W$  a bounded operator from  $\mathfrak{H}_{-}$  to  $\mathfrak{H}_{+}$ . Denote by

$$A = \begin{pmatrix} A_{+} & 0 \\ 0 & -A_{-} \end{pmatrix} \quad \text{and} \quad B = A + V = \begin{pmatrix} A_{+} & W \\ W^{*} & -A_{-} \end{pmatrix}$$

the block operator matrices with respect to the orthogonal decomposition of the Hilbert space  $\mathfrak{H} = \mathfrak{H}_{+} \oplus \mathfrak{H}_{-}$ . Then

$$(1.1) \quad \|\tan 2\Theta\| \leq \frac{2\|V\|}{d}, \quad \text{spec}(\Theta) \subset [0, \pi/4),$$

where  $\Theta$  is the operator angle between the subspaces  $\text{Ran } E_A(\mathbb{R}_{+})$  and  $\text{Ran } E_B(\mathbb{R}_{+})$  and

$$d = \text{dist}(\text{spec}(A_{+}), \text{spec}(-A_{-}))$$

(see, e.g., [8]).

Estimate (1.1) can equivalently be expressed as the following inequality for the norm of the difference of the orthogonal projections  $P = E_A(\mathbb{R}_{+})$  and  $Q = E_B(\mathbb{R}_{+})$ :

$$(1.2) \quad \|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{d} \right),$$

which, in particular, implies the estimate

$$(1.3) \quad \|P - Q\| < \frac{\sqrt{2}}{2}.$$

Independently of the work of Davis and Kahan, inequality (1.3) has been proven by Adamyan and Langer in [1], where the operators  $A_{\pm}$  were allowed to be semibounded. The case  $d = 0$  has been considered in the work [9] by Kostrykin, Makarov, and Motovilov. In particular, it was proven that there is a unique orthogonal projection  $Q$  from the operator interval  $[E_B((0, \infty)), E_B([0, \infty))]$  such that

$$\|P - Q\| \leq \frac{\sqrt{2}}{2},$$

where  $P \in [E_A((0, \infty)), E_A([0, \infty))]$  is the orthogonal projection onto the invariant (not necessary spectral) subspace  $\mathcal{H}_{+} \subset \mathcal{H}$  of the operator  $A$ . A particular case of this result has been

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obtained earlier by Adamyan, Langer, and Tretter, in [2]. Recently, a version of the Tan  $2\Theta$  Theorem for off-diagonal perturbations  $V$  that are relatively bounded with respect to the diagonal operator  $A$  has been proven by Motovilov and Selin in [11].

In the present work we obtain several generalizations of the aforementioned results assuming that the perturbation is given by an off-diagonal symmetric form.

Given a sesquilinear symmetric form  $\mathfrak{a}$  and a self-adjoint involution  $J$  such that the form  $\mathfrak{a}_J[x, y] := \mathfrak{a}[x, Jy]$  is a positive definite and

$$\mathfrak{a}[x, Jy] = \mathfrak{a}[Jx, y],$$

we call a symmetric sesquilinear form  $\mathfrak{v}$  off-diagonal with respect to the orthogonal decomposition  $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$  with  $\mathfrak{H}_\pm = \text{Ran}(I \pm J)$  if

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy].$$

Based on a close relationship between the symmetric form  $\mathfrak{a}[x, y] + \mathfrak{v}[x, y]$  and the sectorial sesquilinear form  $\mathfrak{a}[x, Jy] + i\mathfrak{v}[x, Jy]$  (cf. [11], [13]), under the assumption that the off-diagonal form  $\mathfrak{v}$  is relatively bounded with respect to the form  $\mathfrak{a}_J$ , we prove

- (i) an analog of the First Representation Theorem for block operator matrices defined as not necessarily semibounded quadratic forms,
- (ii) a relative version of the Tan  $2\Theta$  Theorem.

We also provide several versions of the relative Tan  $2\Theta$  Theorem in the case where the form  $\mathfrak{a}$  is semibounded.

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## 2. THE FIRST REPRESENTATION THEOREM FOR OFF-DIAGONAL FORM PERTURBATIONS

To introduce the notation, it is convenient to assume the following hypothesis.

**Hypothesis 2.1.** *Let  $\mathfrak{a}$  be a symmetric sesquilinear form on  $\text{Dom}[\mathfrak{a}]$  in a Hilbert space  $\mathfrak{H}$ . Assume that  $J$  is a self-adjoint involution such that*

$$J \text{Dom}[\mathfrak{a}] = \text{Dom}[\mathfrak{a}].$$

*Suppose that*

$$\mathfrak{a}[Jx, y] = \mathfrak{a}[x, Jy] \quad \text{for all } x, y \in \text{Dom}[\mathfrak{a}_J] = \text{Dom}[\mathfrak{a}],$$

*and that the form  $\mathfrak{a}_J$  given by*

$$\mathfrak{a}_J[x, y] = \mathfrak{a}[x, Jy], \quad x, y \in \text{Dom}[\mathfrak{a}_J] = \text{Dom}[\mathfrak{a}].$$

*is a positive definite closed form. Denote by  $m_\pm$  the greatest lower bound of the form  $\mathfrak{a}_J$  restricted to the subspace*

$$\mathfrak{H}_\pm = \text{Ran}(I \pm J).$$

**Definition 2.2.** *Under Hypothesis 2.1, a symmetric sesquilinear form  $\mathfrak{v}$  on  $\text{Dom}[\mathfrak{v}] \supset \text{Dom}[\mathfrak{a}]$  is said to be off-diagonal with respect to the orthogonal decomposition*

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$$

if

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy], \quad x, y \in \text{Dom}[\mathfrak{a}].$$

If, in addition,

$$(2.1) \quad v_0 := \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_J[x]} < \infty,$$

the form  $\mathfrak{v}$  is said to be an  $\mathfrak{a}$ -bounded off-diagonal form.

**Remark 2.3.** If  $\mathfrak{v}$  is an off-diagonal symmetric form and  $x = x_+ + x_-$  is a unique decomposition of an element  $x \in \text{Dom}[\mathfrak{a}]$  such that  $x_{\pm} \in \mathfrak{H}_{\pm} \cap \text{Dom}[\mathfrak{a}]$ , then

$$(2.2) \quad \mathfrak{v}[x] = 2\text{Re } \mathfrak{v}[x_+, x_-], \quad x \in \text{Dom}[\mathfrak{a}].$$

Moreover, if  $v_0 < \infty$ , then

$$(2.3) \quad |\mathfrak{v}[x]| \leq 2v_0 \sqrt{\mathfrak{a}_J[x_+] \mathfrak{a}_J[x_-]}.$$

*Proof.* To prove (2.2), we use the representation

$$\mathfrak{v}[x] = \mathfrak{v}[x_+ + x_-, x_+ + x_-] = \mathfrak{v}[x_+] + \mathfrak{v}[x_-] + \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+], \quad x \in \text{Dom}[\mathfrak{a}].$$

Since  $\mathfrak{v}$  is an off-diagonal form, one obtains that

$$\mathfrak{v}[x_+] = \mathfrak{v}[x_+, x_+] = \mathfrak{v}[Jx_+, Jx_+] = -\mathfrak{v}[x_+, x_+] = -\mathfrak{v}[x_+] = 0,$$

and similarly  $\mathfrak{v}[x_-] = 0$ . Therefore,

$$\mathfrak{v}[x] = \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+] = 2\text{Re } \mathfrak{v}[x_+, x_-], \quad x \in \text{Dom}[\mathfrak{a}].$$

To prove (2.3), first one observes that

$$\mathfrak{a}_J[x] = \mathfrak{a}_J[x_+] + \mathfrak{a}_J[x_-]$$

and, hence, combining (2.2) and (2.1), one gets the estimate

$$|2\text{Re } \mathfrak{v}[x_+, x_-]| \leq v_0 \mathfrak{a}_J[x] = v_0 (\mathfrak{a}_J[x_+] + \mathfrak{a}_J[x_-]) \quad \text{for all } x_{\pm} \in \mathfrak{H}_{\pm} \cap \text{Dom}[\mathfrak{a}].$$

Hence, for any  $t \geq 0$  (and, therefore, for all  $t \in \mathbb{R}$ ) one gets that

$$v_0 \mathfrak{a}_J[x_+] t^2 - 2|\text{Re } \mathfrak{v}[x_+, x_-]| t + v_0 \mathfrak{a}_J[x_-] \geq 0,$$

which together with (2.2) implies the inequality (2.3).  $\square$

In this setting we present an analog of the First Representation Theorem in the off-diagonal perturbation theory.

**Theorem 2.4.** Assume Hypothesis 2.1. Suppose that  $\mathfrak{v}$  is an  $\mathfrak{a}$ -bounded off-diagonal with respect to the orthogonal decomposition  $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$  symmetric form. On  $\text{Dom}[\mathfrak{b}] = \text{Dom}[\mathfrak{a}]$  introduce the symmetric form

$$\mathfrak{b}[x, y] = \mathfrak{a}[x, y] + \mathfrak{v}[x, y], \quad x, y \in \text{Dom}[\mathfrak{b}].$$

Then

(i) there is a unique self-adjoint operator  $B$  in  $\mathfrak{H}$  such that  $\text{Dom}(B) \subset \text{Dom}[\mathfrak{b}]$  and

$$\mathfrak{b}[x, y] = \langle x, By \rangle \quad \text{for all } x \in \text{Dom}[\mathfrak{b}], \quad y \in \text{Dom}(B).$$

(ii) the operator  $B$  is boundedly invertible and the open interval  $(-m_-, m_+) \ni 0$  belongs to its resolvent set.

*Proof.* (i). Given  $\mu \in (-m_-, m_+)$ , on  $\text{Dom}[\mathfrak{a}_\mu] = \text{Dom}[\mathfrak{a}]$  introduce the positive closed form  $\mathfrak{a}_\mu$  by

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu \langle x, Jy \rangle, \quad x, y \in \text{Dom}[\mathfrak{a}_\mu],$$

and denote by  $\mathfrak{H}_{\mathfrak{a}_\mu}$  the Hilbert space  $\text{Dom}[\mathfrak{a}_\mu]$  equipped with the inner product  $\langle \cdot, \cdot \rangle_\mu = \mathfrak{a}_\mu[\cdot, \cdot]$ . We remark that the norms  $\|\cdot\|_\mu = \sqrt{\mathfrak{a}_\mu[\cdot]}$  on  $\mathfrak{H}_{\mathfrak{a}_\mu} = \text{Dom}[\mathfrak{a}_\mu]$  are obviously equivalent. Since  $\mathfrak{v}$  is  $\mathfrak{a}$ -bounded, one concludes then that

$$v_\mu := \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]} < \infty, \quad \text{for all } \mu \in (-m_-, m_+).$$

Along with the off-diagonal form  $\mathfrak{v}$ , introduce a dual form  $\mathfrak{v}'$  by

$$\mathfrak{v}'[x, y] = i\mathfrak{v}[x, Jy], \quad x, y \in \text{Dom}[\mathfrak{a}].$$

We claim that  $\mathfrak{v}'$  is an  $\mathfrak{a}$ -bounded off-diagonal symmetric form. It suffices to show that

$$v_\mu = v'_\mu < \infty, \quad \mu \in (-m_-, m_+),$$

where

$$(2.4) \quad v'_\mu := \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}'[x]|}{\mathfrak{a}_\mu[x]}.$$

Indeed, let  $x = x_+ + x_-$  be a unique decomposition of an element  $x \in \text{Dom}[\mathfrak{a}]$  such that  $x_\pm \in \mathfrak{H}_\pm \cap \text{Dom}[\mathfrak{a}]$ . By Remark 2.3,

$$\mathfrak{v}[x] = \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+] = 2\text{Re } \mathfrak{v}[x_+, x_-], \quad x \in \text{Dom}[\mathfrak{a}].$$

In a similar way (since the form  $\mathfrak{v}'$  is obviously off-diagonal) one gets that

$$\begin{aligned} \mathfrak{v}'[x] &= i\mathfrak{v}[x_+ + x_-, J(x_+ + x_-)] = i\mathfrak{v}'[x_+] - i\mathfrak{v}'[x_-] - i\mathfrak{v}[x_+, x_-] + i\mathfrak{v}[x_-, x_+] \\ &= -i\mathfrak{v}[x_+, x_-] + \overline{i\mathfrak{v}[x_+, x_-]} = 2\text{Im } \mathfrak{v}[x_+, x_-], \quad x \in \text{Dom}[\mathfrak{a}]. \end{aligned}$$

Clearly, from (2.4) it follows that

$$\begin{aligned} v'_\mu &= 2 \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\text{Im } \mathfrak{v}[x_+, x_-]|}{\mathfrak{a}_\mu[x]} = 2 \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\text{Re } \mathfrak{v}[x_+, x_-]|}{\mathfrak{a}_\mu[x]} = v_\mu, \\ &\mu \in (-m_-, m_+), \end{aligned}$$

which completes the proof of the claim.

Next, on  $\text{Dom}[\mathfrak{t}_\mu] = \text{Dom}[\mathfrak{a}]$  introduce the sesquilinear form

$$\mathfrak{t}_\mu := \mathfrak{a}_\mu + i\mathfrak{v}', \quad \mu \in (-m_-, m_+).$$

Since the form  $\mathfrak{a}_\mu$  is positive definite and the form  $\mathfrak{v}'$  is an  $\mathfrak{a}_\mu$ -bounded symmetric form, the form  $\mathfrak{t}$  is a closed sectorial form with the vertex 0 and semi-angle

$$(2.5) \quad \theta_\mu = \arctan(v'_\mu) = \arctan(v_\mu).$$

Let  $T_\mu$  be a unique  $m$ -sectorial operator associated with the form  $\mathfrak{t}_\mu$ . Introduce the operator

$$B_\mu = JT_\mu \quad \text{on} \quad \text{Dom}(B_\mu) = \text{Dom}(T_\mu), \quad \mu \in (-m_-, m_+).$$

One obtains that

$$\begin{aligned} \langle x, B_\mu y \rangle &= \langle x, JT_\mu y \rangle = \langle Jx, T_\mu y \rangle = \mathfrak{a}_\mu[Jx, y] + i\mathfrak{v}'[Jx, y] \\ (2.6) \quad &= \mathfrak{a}[x, y] - \mu \langle Jx, Jy \rangle + i^2 \mathfrak{v}[Jx, Jy] \\ &= \mathfrak{a}[x, y] - \mu \langle x, y \rangle + \mathfrak{v}[x, y], \end{aligned}$$

for all  $x \in \text{Dom}[\mathfrak{a}]$ ,  $y \in \text{Dom}(B_\mu) = \text{Dom}(T_\mu)$ . In particular,  $B_\mu$  is a symmetric operator on  $\text{Dom}(B_\mu)$ , since the forms  $\mathfrak{a}$  and  $\mathfrak{v}$  are symmetric, and  $\text{Dom}(B_\mu) = \text{Dom}(T_\mu) \subset \text{Dom}[\mathfrak{a}]$ .

For the real part of the form  $t_\mu$  is positive definite with a positive lower bound, the operator  $T_\mu$  has a bounded inverse. This implies that the operator  $B_\mu = JT_\mu$  has a bounded inverse and, therefore, the symmetric operator  $B_\mu$  is self-adjoint on  $\text{Dom}(B_\mu)$ .

As an immediate consequence, one concludes (put  $\mu = 0$ ) that the self-adjoint operator  $B := B_0$  is associated with the symmetric form  $\mathfrak{b}$  and that  $\text{Dom}(B) \subset \text{Dom}[\mathfrak{a}]$ .

To prove uniqueness, assume that  $B'$  is a self-adjoint operator associated with the form  $\mathfrak{b}$ . Then for all  $x \in \text{Dom}(B)$  and all  $y \in \text{Dom}(B')$  one gets that

$$\langle x, B'y \rangle = \mathfrak{b}[x, y] = \overline{\mathfrak{b}[y, x]} = \overline{\langle y, Bx \rangle} = \langle Bx, y \rangle,$$

which means that  $B = (B')^* = B'$ .

(ii). From (2.6) one concludes that the self-adjoint operator  $B_\mu + \mu I$  is associated with the form  $\mathfrak{b}$  and, hence, by the uniqueness

$$B_\mu = B - \mu I \quad \text{on} \quad \text{Dom}(B_\mu) = \text{Dom}(B).$$

Since  $B_\mu$  has a bounded inverse for all  $\mu \in (m_-, m_+)$ , so does  $B - \mu I$  which means that the interval  $(-m_-, m_+)$  belongs to the resolvent set of the operator  $B_0$ .  $\square$

**Remark 2.5.** In the particular case  $\mathfrak{v} = 0$ , from Theorem 2.4 it follows that there exists a unique self-adjoint operator  $A$  associated with the form  $\mathfrak{a}$ .

For a different, more constructive proof of Theorem 2.4 as well as for the history of the subject we refer to our work [4].

**Remark 2.6.** For the part (i) of Theorem 2.4 to hold it is not necessary to require that the form  $\mathfrak{a}_J$  in Hypothesis 2.1 is positive definite. It is sufficient to assume that  $\mathfrak{a}_J$  is a semi-bounded from below closed form (see, e.g., [12]).

### 3. THE TAN $2\Theta$ THEOREM

The main result of this work provides a sharp upper bound for the angle between the positive spectral subspaces  $\text{Ran } E_A(\mathbb{R}_+)$  and  $\text{Ran } E_B(\mathbb{R}_+)$  of the operators  $A$  and  $B$  respectively.

**Theorem 3.1.** Assume Hypothesis 2.1 and suppose that  $\mathfrak{v}$  is off-diagonal with respect to the decomposition  $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ . Let  $A$  be a unique self-adjoint operator associated with the form  $\mathfrak{a}$  and  $B$  the self-adjoint operator associated with the form  $\mathfrak{b} = \mathfrak{a} + \mathfrak{v}$  referred to in Theorem 2.4.

Then the norm of the difference of the spectral projections  $P = E_A(\mathbb{R}_+)$  and  $Q = E_B(\mathbb{R}_+)$  satisfies the estimate

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan v \right) < \frac{\sqrt{2}}{2},$$

where

$$v = \inf_{\mu \in (-m_-, m_+)} v_\mu = \inf_{\mu \in (-m_-, m_+)} \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]},$$

with

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu \langle x, Jy \rangle, \quad x, y \in \text{Dom}[\mathfrak{a}_\mu] = \text{Dom}[\mathfrak{a}].$$

The proof of Theorem 3.1 uses the following result borrowed from [14].

**Proposition 3.2.** Let  $T$  be an  $m$ -sectorial operator of semi-angle  $\theta < \pi/2$ . Let  $T = U|T|$  be its polar decomposition. If  $U$  is unitary, then the unitary operator  $U$  is sectorial with semi-angle  $\theta$ .

**Remark 3.3.** We note that for a bounded sectorial operator  $T$  with a bounded inverse the statement is quite simple. Due to the equality

$$\langle x, Tx \rangle = \langle |T|^{-1/2}y, U|T|^{1/2}y \rangle = \langle y, |T|^{-1/2}U|T|^{1/2}y \rangle, \quad y = |T|^{1/2}x,$$

the operators  $T$  and  $|T|^{-1/2}U|T|^{1/2}$  are sectorial with the semi-angle  $\theta$ . The resolvent sets of the operators  $|T|^{-1/2}U|T|^{1/2}$  and  $U$  coincide. Therefore, since  $U$  is unitary, it follows that  $U$  is sectorial with semi-angle  $\theta$ .

*Proof of Theorem 3.1.* Given  $\mu \in (-m_-, m_+)$ , let  $T_\mu = U_\mu|T_\mu|$  be the polar decomposition of the sectorial operator  $T_\mu$  with vertex 0 and semi-angle  $\theta_\mu$ , with

$$(3.1) \quad \theta_\mu = \arctan(v_\mu)$$

(as in the proof of Theorem 2.4 (cf. (2.5)). Since  $B_\mu = JT_\mu$ , one concludes that

$$|T_\mu| = |B_\mu| \quad \text{and} \quad U_\mu = J^{-1} \text{sign}(B_\mu).$$

Since  $T_\mu$  is a sectorial operator with semi-angle  $\theta_\mu$ , by a result in [14] (see Proposition 3.2), the unitary operator  $U_\mu$  is sectorial with vertex 0 and semi-angle  $\theta_\mu$  as well. Therefore, applying the spectral theorem for the unitary operator  $U_\mu$  from (3.1) one obtains the estimate

$$\|J - \text{sign}(B_\mu)\| = \|I - J^{-1} \text{sign}(B_\mu)\| = \|I - U_\mu\| \leq 2 \sin\left(\frac{1}{2} \arctan v_\mu\right).$$

Since the open interval  $(-m_-, m_+)$  belongs to the resolvent set of the operator  $B = B_0$ , the involution  $\text{sign}(B_\mu)$  does not depend on  $\mu \in (-m_-, m_+)$  and hence one concludes that

$$\text{sign}(B_\mu) = \text{sign}(B_0) = \text{sign}(B), \quad \mu \in (-m_-, m_+).$$

Therefore,

$$(3.2) \quad \|P - Q\| = \frac{1}{2} \|J - \text{sign}(B)\| = \frac{1}{2} \|J - \text{sign}(B_\mu)\| \leq \sin\left(\frac{1}{2} \arctan v_\mu\right)$$

and, hence, since  $\mu \in (-m_-, m_+)$  has been chosen arbitrarily, from (3.2) it follows that

$$\|P - Q\| \leq \inf_{\mu \in (-m_-, m_+)} \sin\left(\frac{1}{2} \arctan v_\mu\right) \leq \sin\left(\frac{1}{2} \arctan v\right).$$

The proof is complete.  $\square$

As a consequence, we have the following result that can be considered a geometric variant of the Birman-Schwinger principle for the off-diagonal form-perturbations.

**Corollary 3.4.** *Assume Hypothesis 2.1 and suppose that  $\mathfrak{v}$  is off-diagonal. Then the form  $\mathfrak{a}_J + \mathfrak{v}$  is positive definite if and only if the  $\mathfrak{a}_J$ -relative bound (2.1) of  $\mathfrak{v}$  does not exceed one. In this case*

$$\|P - Q\| \leq \sin\left(\frac{\pi}{8}\right),$$

where  $P$  and  $Q$  are the spectral projections referred to in Theorem 3.1.

*Proof.* Since  $\mathfrak{v}$  is an  $\mathfrak{a}$ -bounded form, one concludes that there exists a self-adjoint bounded operator  $\mathcal{V}$  in the Hilbert space  $\text{Dom}[\mathfrak{a}]$  such that

$$v[x, y] = \mathfrak{a}_J[x, \mathcal{V}y], \quad x, y \in \text{Dom}[\mathfrak{a}].$$

Since  $\mathfrak{v}$  is off-diagonal, the numerical range of  $\mathcal{V}$  coincides with the symmetric about the origin interval  $[-\|\mathcal{V}\|, \|\mathcal{V}\|]$ . Therefore, one can find a sequence  $\{x_n\}_{n=1}^\infty$  in  $\text{Dom}[\mathfrak{a}]$  such that

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{v}[x_n]}{\mathfrak{a}_J[x_n]} = -\|\mathcal{V}\|,$$

which proves that  $\|\mathcal{V}\| \leq 1$  if and only if the form  $\mathfrak{a}_J + \mathfrak{v}$  is positive definite. If it is the case, applying Theorem 3.1, one obtains the inequality

$$\|P - Q\| \leq \sin\left(\frac{1}{2} \arctan(\|\mathcal{V}\|)\right) \leq \sin\left(\frac{\pi}{8}\right)$$

which completes the proof.  $\square$

**Remark 3.5.** We remark that in accordance with the Birman-Schwinger principle, for the form  $\mathfrak{a}_J + \mathfrak{v}$  to have negative spectrum it is necessary that the  $\mathfrak{a}_J$ -relative bound  $\|\mathcal{V}\|$  of the perturbation  $\mathfrak{v}$  is greater than one. As Corollary 3.4 shows, in the off-diagonal perturbation theory this condition is also sufficient.

#### 4. TWO SHARP ESTIMATES IN THE SEMIBOUNDED CASE

In this section we will be dealing with the case of off-diagonal form-perturbations of a semi-bounded operator.

**Hypothesis 4.1.** Assume that  $A$  is a self-adjoint semi-bounded from below operator. Suppose that  $A$  has a bounded inverse. Assume, in addition, that the following conditions hold:

- (i) The spectral condition. An open finite interval  $(\alpha, \beta)$  belongs to the resolvent set of the operator  $A$ . We set

$$\Sigma_- = \text{spec}(A) \cap (-\infty, \alpha] \quad \text{and} \quad \Sigma_+ = \text{spec}(A) \cap [\beta, \infty).$$

- (ii) Boundedness. The sesquilinear form  $\mathfrak{v}$  is symmetric on  $\text{Dom}[\mathfrak{v}] \supset \text{Dom}(|A|^{1/2})$  and

$$(4.1) \quad v := \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\| |A|^{1/2} x \|^2} < \infty.$$

- (iii) Off-diagonality. The sesquilinear form  $\mathfrak{v}$  is off-diagonal with respect to the orthogonal decomposition  $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ , with

$$\mathfrak{H}_+ = \text{Ran } E_A((\beta, \infty)) \quad \text{and} \quad \mathfrak{H}_- = \text{Ran } E_A((-\infty, \alpha)).$$

That is,

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy], \quad x, y \in \text{Dom}[\mathfrak{a}],$$

where the self-adjoint involution  $J$  is given by

$$(4.2) \quad J = E_A((\beta, \infty)) - E_A((-\infty, \alpha)).$$

Let  $\mathfrak{a}$  be the closed form represented by the operator  $A$ . A direct application of Theorem 2.4 shows that under Hypothesis 4.1 there is a unique self-adjoint boundedly invertible operator  $B$  associated with the form

$$\mathfrak{b} = \mathfrak{a} + \mathfrak{v}.$$

Under Hypothesis 4.1 we distinguish two cases (see Fig. 1 and 2).

**Case I.** Assume that  $\alpha < 0$  and  $\beta > 0$ . Set

$$d_+ = \text{dist}(\inf(\Sigma_+), 0) \quad \text{and} \quad d_- = \text{dist}(\inf(\Sigma_-), 0)$$

and suppose that  $d_+ > d_-$ .

**Case II.** Assume that  $\alpha, \beta > 0$ . Set

$$d_+ = \text{dist}(\inf(\Sigma_+), 0) \quad \text{and} \quad d_- = \text{dist}(\sup(\Sigma_-), 0).$$

As it follows from the definition of the quantities  $d_{\pm}$ , the sum  $d_- + d_+$  coincides with the distance between the lower edges of the spectral components  $\Sigma_+$  and  $\Sigma_-$  in Case I, while in Case II the difference  $d_+ - d_-$  is the distance from the lower edge of  $\Sigma_+$  to the upper edge of the spectral component  $\Sigma_-$ . Therefore,  $d_+ - d_-$  coincides with the length of the spectral gap  $(\alpha, \beta)$  of the operator  $A$  in latter case.

We remark that the condition  $d_+ > d_-$  required in Case I, holds only if the length of the convex hull of negative spectrum  $\Sigma_-$  of  $A$  does not exceed the one of the spectral gap  $(\alpha, \beta) = (\sup(\Sigma_-), \inf(\Sigma_+))$ .

Now we are prepared to state a relative version of the Tan  $2\Theta$  Theorem in the case where the unperturbed operator is semi-bounded or even positive.

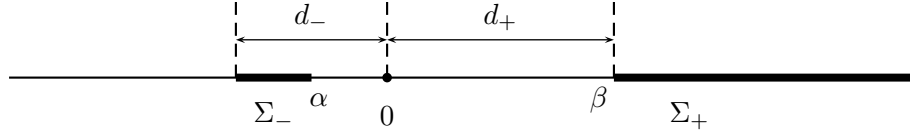


FIG. 1. The spectrum of the unperturbed sign-indefinite semibounded invertible operator  $A$  in Case I.

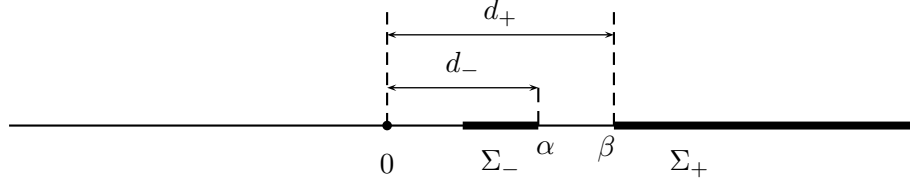


FIG. 2. The spectrum of the unperturbed strictly positive operator  $A$  with a gap in its spectrum in Case II.

**Theorem 4.2.** *In either Cases I or II, introduce the spectral projections*

$$(4.3) \quad P = E_A((-\infty, \alpha]) \quad \text{and} \quad Q = E_B((-\infty, \alpha])$$

*of the operators  $A$  and  $B$  respectively.*

*Then the norm of the difference of  $P$  and  $Q$  satisfies the estimate*

$$(4.4) \quad \|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \left[ 2 \frac{v}{\delta} \right] \right) < \frac{\sqrt{2}}{2},$$

*where*

$$(4.5) \quad \delta = \frac{1}{\sqrt{d_+ d_-}} \begin{cases} d_+ + d_- & \text{in Case I,} \\ d_+ - d_- & \text{in Case II,} \end{cases}$$

*and  $v$  stands for the relative bound of the off-diagonal form  $\mathfrak{v}$  (with respect to  $\mathfrak{a}$ ) given by (4.1).*

*Proof.* We start with the remark that the form  $\mathfrak{a} - \mu$ , where  $\mathfrak{a}$  is the form of  $A$ , satisfies Hypothesis 2.1 with  $J$  given by (4.2). Set

$$\mathfrak{a}_\mu = (\mathfrak{a} - \mu)_J, \quad \mu \in (\alpha, \beta),$$

that is,

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu[x, Jy], \quad x, y \in \text{Dom}[\mathfrak{a}].$$

Notice that  $\mathfrak{a}_\mu$  is a strictly positive closed form represented by the operators  $JA - J\mu = |A| - \mu J$  and  $JA - \mu J = |A - \mu I|$  in Cases I and II, respectively.

Since  $\mathfrak{v}$  is off-diagonal, from Theorem 3.1 it follows that

$$(4.6) \quad \|E_{A-\mu I}(\mathbb{R}_+) - E_{B-\mu I}(\mathbb{R}_+)\| \leq \sin \left( \frac{1}{2} \arctan v_\mu \right) \quad \text{for all} \quad \mu \in (\alpha, \beta),$$

with

$$(4.7) \quad v_\mu =: \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]}.$$

Since  $\mathfrak{v}$  is off-diagonal, by Remark 2.3 one gets the estimate

$$|\mathfrak{v}[x]| \leq 2v_0 \sqrt{\mathfrak{a}_0[x_+] \mathfrak{a}_0[x_-]}, \quad x \in \text{Dom}[\mathfrak{a}],$$



where  $x = x_+ + x_-$  is a unique decomposition of the element  $x \in \text{Dom}[\mathfrak{a}]$  with

$$x_{\pm} \in \mathfrak{H}_{\pm} \cap \text{Dom}[\mathfrak{a}].$$

Thus, in these notations, taking into account that

$$v_0 = v,$$

where  $v$  is given by (4.1), one gets the bound

$$(4.8) \quad v_{\mu} \leq 2v \sup_{0 \neq x \in \text{Dom}[\mathfrak{a}]} \frac{\sqrt{\mathfrak{a}_0[x_+] \mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\mu}[x]}.$$

Since  $\mathfrak{a}_{\mu}$  is represented by  $JA - J\mu = |A| - \mu J$  and  $JA - \mu J = |A - \mu I|$  in Cases I and II, respectively, one observes that

$$(4.9) \quad \mathfrak{a}_{\mu}[x] = \begin{cases} \mathfrak{a}_0[x_+] - \mu \|x_+\|^2 + \mathfrak{a}_0[x_-] + \mu \|x_-\|^2, & \text{in Case I,} \\ \mathfrak{a}_0[x_+] - \mu \|x_+\|^2 - \mathfrak{a}_0[x_-] + \mu \|x_-\|^2, & \text{in Case II.} \end{cases}$$

Introducing the elements  $y_{\pm} \in \mathfrak{H}_{\pm}$ ,

$$y_{\pm} := \begin{cases} (|A| \mp \mu I)^{1/2} x_{\pm}, & \text{in Case I,} \\ \pm(A - \mu I)^{1/2} x_{\pm}, & \text{in Case II,} \end{cases}$$

and taking into account (4.9), one obtains the representation

$$\frac{\sqrt{\mathfrak{a}_0[x_+] \mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\mu}[x]} = \frac{\| |A|^{1/2} (|A| - \mu I)^{-1/2} y_+ \| \| |A|^{1/2} (-A + \mu I)^{-1/2} y_- \|}{\|y_+\|^2 + \|y_-\|^2},$$

valid in both Cases I and II. Using the elementary inequality

$$\|y_+\| \|y_-\| \leq \frac{1}{2} (\|y_+\|^2 + \|y_-\|^2),$$

one arrives at the following bound

$$(4.10) \quad \frac{\sqrt{\mathfrak{a}_0[x_+] \mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\mu}[x]} \leq \frac{1}{2} \| |A|^{1/2} (|A| - \mu I)^{-1/2} |_{\mathfrak{H}_+} \| \cdot \| |A|^{1/2} (-A + \mu I)^{-1/2} |_{\mathfrak{H}_-} \|.$$

It is easy to see that

$$(4.11) \quad \| |A|^{1/2} (|A| - \mu I)^{-1/2} |_{\mathfrak{H}_+} \| \leq \frac{\sqrt{d_+}}{\sqrt{d_+ - \mu}} \quad \mu \in (\alpha, \beta), \quad \text{in Cases I and II,}$$

while

$$(4.12) \quad \| |A|^{1/2} (-A + \mu I)^{-1/2} |_{\mathfrak{H}_-} \| \leq \begin{cases} \frac{\sqrt{d_-}}{\sqrt{d_- + \mu}}, & \mu \in (0, \beta), \quad \text{in Case I,} \\ \frac{\sqrt{d_-}}{\sqrt{\mu - d_-}}, & \mu \in (\alpha, \beta), \quad \text{in Case II.} \end{cases}$$

Choosing  $\mu = \frac{d_+ - d_-}{2} > 0$  in Case I (recall that  $d_+ > d_-$  by the hypothesis) and  $\mu = \frac{d_+ + d_-}{2}$  in Case II, and combining (4.10), (4.11), (4.12), one gets the estimates

$$\frac{\sqrt{\mathfrak{a}_0[x_+] \mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\frac{d_+ - d_-}{2}}[x]} \leq \frac{\sqrt{d_+ d_-}}{d_+ + d_-} \quad \text{in Case I}$$

and

$$\frac{\sqrt{\mathfrak{a}_0[x_+] \mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\frac{d_+ + d_-}{2}}[x]} \leq \frac{\sqrt{d_+ + d_-}}{d_+ - d_-} \quad \text{in Case II.}$$

Hence, from (4.8) it follows that

$$v \frac{d_+ - d_-}{2} \leq 2v \frac{\sqrt{d_+ d_-}}{d_+ + d_-} \quad \text{in Case I}$$

and

$$v \frac{d_+ + d_-}{2} \leq 2v \frac{\sqrt{d_+ d_-}}{d_+ - d_-} \quad \text{in Case II.}$$

Applying (4.6), one gets the norm estimates

$$(4.13) \quad \|E_{A - \frac{d_+ - d_-}{2}I}(\mathbb{R}_+) - E_{B - \frac{d_+ - d_-}{2}I}(\mathbb{R}_+)\| \leq \sin \left( \frac{1}{2} \arctan \left[ 2 \frac{\sqrt{d_+ d_-}}{d_+ + d_-} v \right] \right)$$

in Case I and

$$(4.14) \quad \|E_{A - \frac{d_+ + d_-}{2}I}(\mathbb{R}_+) - E_{B - \frac{d_+ + d_-}{2}I}(\mathbb{R}_+)\| \leq \sin \left( \frac{1}{2} \arctan \left[ 2 \frac{\sqrt{d_+ d_-}}{d_+ - d_-} v \right] \right)$$

in Case II. It remains to observe that  $\|P - Q\|$ , where the spectral projections  $P$  and  $Q$  are given by (4.3), coincides with the left hand side of (4.13) and (4.14) in Case I and Case II, respectively.

The proof is complete.  $\square$

**Remark 4.3.** We remark that the quantity  $\delta$  given by (4.5) coincides with the relative distance (with respect to the origin) between the lower edges of the spectral components  $\Sigma_+$  and  $\Sigma_-$  in Case I and it has the meaning of the relative length (with respect to the origin) of the spectral gap  $(d_-, d_+)$  in Case II.

For the further properties of the relative distance and various relative perturbation bounds we refer to the paper [10] and references quoted therein.

We also remark that in Case II, i.e., in the case of a positive operator  $A$ , the bound (4.4) directly improves a result obtained in [6], the relative  $\sin \Theta$  Theorem, that in the present notations is of the form

$$\|P - Q\| \leq \frac{v}{\delta}.$$

We conclude our exposition with considering an example of a  $2 \times 2$  numerical matrix that shows that the main results obtained above are sharp.

**Example 4.4.** Let  $\mathfrak{H}$  be the two-dimensional Hilbert space  $\mathfrak{H} = \mathbb{C}^2$ ,  $\alpha < \beta$  and  $w \in \mathbb{C}$ .

We set

$$A = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \quad V = \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let  $\mathfrak{v}$  be the symmetric form represented by (the operator)  $V$ .

Clearly, the form  $\mathfrak{v}$  satisfy Hypothesis 4.1 with the relative bound  $v$  given by

$$v = \frac{|w|}{\sqrt{|\alpha\beta|}},$$

provided that  $\alpha, \beta \neq 0$ . Since  $VJ = -JV$ , the form  $\mathfrak{v}$  is off-diagonal with respect to the orthogonal decomposition  $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ .

In order to illustrate our results, denote by  $B$  the self-adjoint matrix associated with the form  $\mathfrak{a} + \mathfrak{v}$ , that is,

$$B = A + V = \begin{pmatrix} \beta & w \\ w^* & \alpha \end{pmatrix}.$$

Denote by  $P$  the orthogonal projection associated with the eigenvalue  $\alpha$  of the matrix  $A$ , and by  $Q$  the one associated with the lower eigenvalue of the matrix  $B$ .

It is well known (and easy to see) that the classical Davis-Kahan Tan  $2\Theta$  theorem (1.2) is exact in the case of  $2 \times 2$  numerical matrices. In particular, the norm of the difference of  $P$  and  $Q$  can be computed explicitly

$$(4.15) \quad \|P - Q\| = \sin \left( \frac{1}{2} \arctan \left[ \frac{2|w|}{\beta - \alpha} \right] \right).$$

Since, in the case in question,

$$(4.16) \quad v_\mu = \sup_{0 \neq x \in \text{Dom}[a]} \frac{|\mathfrak{b}[x]|}{\mathfrak{a}_\mu[x]} = \frac{|w|}{\sqrt{(\beta - \mu)(\mu - \alpha)}}, \quad \mu \in (\alpha, \beta),$$

from (4.16) it follows that

$$\inf_{\mu \in (\alpha, \beta)} v_\mu = \frac{2|w|}{\beta - \alpha}$$

(with the infimum attained at the point  $\mu = \frac{\alpha + \beta}{2}$ ).

Therefore, the result of the relative tan  $2\Theta$  Theorem 3.1 is sharp.

It is easy to see that if  $\alpha < 0 < \beta$  (Case I), then the equality (4.15) can also be rewritten in the form

$$(4.17) \quad \|P - Q\| = \sin \left( \frac{1}{2} \arctan \left[ 2 \frac{\sqrt{d_+ d_-}}{d_+ + d_-} v \right] \right),$$

where  $d_+ = \beta$ ,  $d_- = -\alpha$  and  $v = \frac{|w|}{\sqrt{|\alpha|\beta}}$ .

If  $0 < \alpha < \beta$  (Case II), the equality (4.15) can be rewritten as

$$(4.18) \quad \|P - Q\| = \sin \left( \frac{1}{2} \arctan \left[ 2 \frac{\sqrt{d_+ d_-}}{d_+ - d_-} v \right] \right),$$

with  $d_+ = \beta$ ,  $d_- = \alpha$ , and  $v = \frac{|w|}{\sqrt{\alpha\beta}}$ .

The representations (4.17) and (4.18) show that the estimate (4.4) becomes equality in the case of  $2 \times 2$  numerical matrices and, therefore, the results of Theorem 4.2 are sharp.

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L. GRUBIŠIĆ, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA 30, 10000 ZAGREB, CROATIA

*E-mail address:* luka.grubisic@math.hr

V. KOSTRYKIN, FB 08 - INSTITUT FÜR MATHEMATIK, JOHANNES GUTENBERG-UNIVERSITÄT MAINZ, STAUDINGER WEG 9, D-55099 MAINZ, GERMANY

*E-mail address:* kostrykin@mathematik.uni-mainz.de

K. A. MAKAROV, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

*E-mail address:* makarovk@missouri.edu

K. VESELIĆ, FAKULTÄT FÜR MATHEMATIK UND INFORMATIK, FERNUNIVERSITÄT HAGEN, POSTFACH 940, D-58084 HAGEN, GERMANY

*E-mail address:* kresimir.veselic@fernuni-hagen.de